

正六角形格子上的 1:2 共鳴

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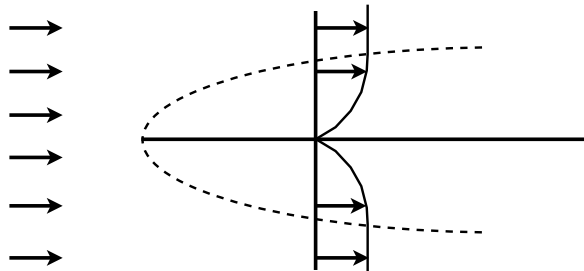
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共鳴三角形による流れパターンの複雑化の例

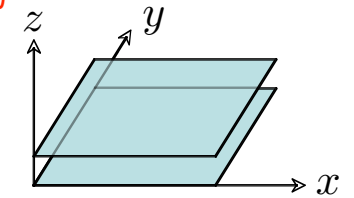
C (Crainik) - タイプ: 2次元トルミン・シュリヒティング波からの3次元分化 (ブラジウス境界層)



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空間的共鳴

$$\psi(\mathbf{x}, t) = \sum_j A_j(t) \phi_j(z) e^{i(k_j^x x + k_j^y y - \omega_j t)}$$



共鳴条件:

$$\sum_n p_n \mathbf{k}_n = \mathbf{0} \text{ and } \sum_n p_n \omega_n = 0 \text{ for } \mathbf{k}_j = (k_j^x, k_j^y) \text{ and } p_n \in \mathbb{Z}$$

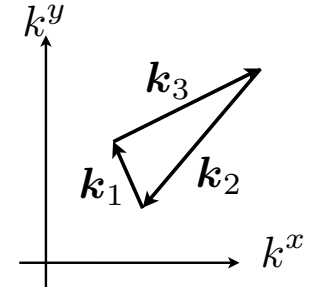
$|p_1| + |p_2| + |p_3| = 3$: 共鳴三角形

$$\dot{A}_1 = \sigma_1 A_1 + \kappa_1 \bar{A}_2 \bar{A}_3 + O(|A_j|^3),$$

$$\dot{A}_2 = \sigma_2 A_2 + \kappa_2 \bar{A}_3 \bar{A}_1 + O(|A_j|^3),$$

$$\dot{A}_3 = \sigma_3 A_3 + \kappa_3 \bar{A}_1 \bar{A}_2 + O(|A_j|^3)$$

$\sigma_n, \kappa_n \in \mathbb{C}$



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1:2 Hopf/定常モード相互作用

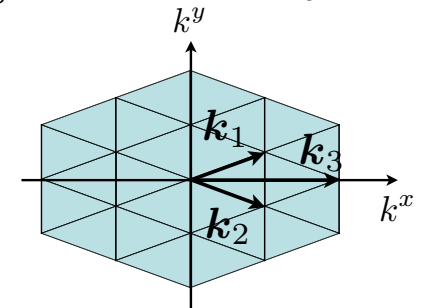
$k_1^x = k_2^x, k_3^x = 2k_1^x, k_1^y = -k_2^y$, and $k_3^y = 0, \omega_1 = -\omega_2$ and $\omega_3 = 0$:

$$\dot{A}_1 = \sigma_1 A_1 + \kappa_1 \bar{A}_2 A_3 + O(|A_j|^3),$$

$$\dot{A}_2 = \bar{\sigma}_1 A_2 + \bar{\kappa}_1 A_3 \bar{A}_1 + O(|A_j|^3),$$

$$\dot{A}_3 = \sigma_3 A_3 + \kappa_3 A_1 A_2 + O(|A_j|^3)$$

$\sigma_1, \kappa_1, \in \mathbb{C}, \sigma_3, \kappa_3 \in \mathbb{R}$



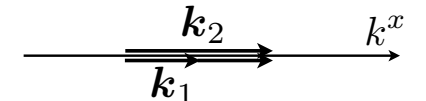
1:2 定常モード間相互作用

$k_1^x = k_2^x, k_3 = 2k_1, k_1^y = k_2^y = k_3^y = 0$, and $\omega_1 = \omega_2 = \omega_3 = 0$:

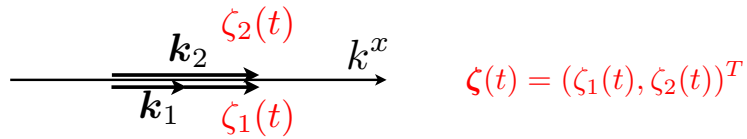
$$\dot{A}_1 = \sigma_1 A_1 + \kappa_1 \bar{A}_1 A_2 + O(|A_j|^3),$$

$$\dot{A}_2 = \sigma_2 A_2 + \kappa_2 A_1^2 + O(|A_j|^3)$$

$\sigma_1, \sigma_2, \kappa_1, \kappa_2 \in \mathbb{R}$



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O(2) 対称下での 1:2 定常モード相互作用

$\zeta_1(t)$ と $\zeta_2(t)$ に対する方程式のもつ対称性: Γ

$$\frac{d}{dt}\zeta = g(\zeta, \lambda) \rightarrow \frac{d}{dt}(\gamma\zeta) = g(\gamma\zeta, \lambda) \text{ for all } \gamma \in \Gamma$$

$$\Leftrightarrow g(\gamma\zeta, \lambda) = \gamma g(\zeta, \lambda) \text{ for all } \gamma \in \Gamma: \Gamma\text{-同変性}$$

Γ -不変実多項式 $f(\zeta)$:

$$f(\gamma\zeta) = f(\zeta) \text{ for all } \gamma \in \Gamma$$

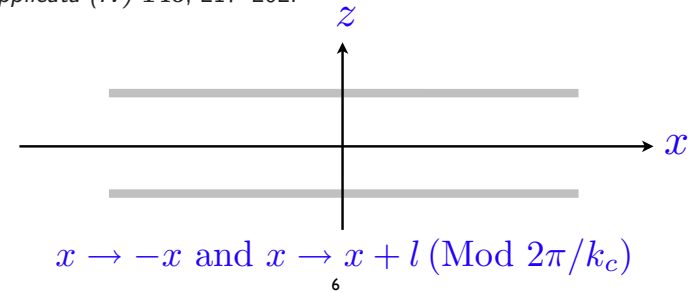
Γ -同変ベクトル場によって生成された振幅方程式:

$$\frac{d}{dt}\zeta = g(\zeta, \lambda) = \sum_j f_j(\zeta)g_j(\zeta, \lambda)$$

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定常解の分類

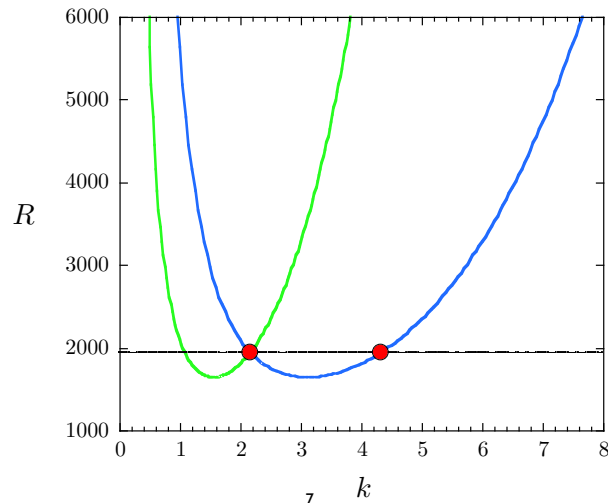
1. Dangelmayr, G. 1986 Steady state mode interactions in presence of O(2) symmetry. *Dyn. Stab. Syst.* 1, 159–185.
2. Buzano, E. & Russo, A. 1987 Bifurcation problems with O(2)⊕Z₂ symmetry and the buckling of a cylindrical shell. *Annali di Matematica Pura ed Applicata (IV)* 146, 217–262.



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O(2) 対称下での 1:2 定常モード相互作用

レイリー・ベナール問題



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O(2) 対称下での 1:2 定常モード相互作用

レイリー・ベナール問題における 1:2 定常モード相互作用

1. Busse, F. H. & Or, A. C. 1986 Subharmonic and asymmetric convection rolls. *ZAMP* 37, 608–623.
2. Armbruster, D. 1987 O(2)-symmetric bifurcation theory for convection rolls. *Physica* 27D, 433–439.
3. Manogg, G. & Metzener, P. 1994 Strong resonance in two-dimensional non-Boussinesq convection. *Phys. Fluids* 6, 2944–2955.
4. Mercader, I., Prat, J. & Knobloch, E. 2001 The 1:2 mode interaction in Rayleigh-Bénard convection with weakly broken midplane symmetry. *Int. J. Bifurcation and Chaos* 11, 27–41.
5. Mercader, I., Prat, J. & Knobloch, E. 2002a The 1:2 mode interaction in Rayleigh-Bénard convection with and without Boussinesq symmetry. *Int. J. Bifurcation and Chaos* 12, 281–308.

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O(2) 対称性の下での 1:2 定常モード相互作用

$$\psi(\mathbf{x}, t) = \zeta_1(t)\phi_1(z) e^{ik_0x} + \zeta_2(t)\phi_2(z) e^{2ik_0x} + c.c. + \dots$$

O(2) = SO(2) × Z₂(κ)

SO(2): $x \rightarrow x + d \pmod{2\pi/k_0}$

$$\Leftrightarrow (\zeta_1, \zeta_2) \rightarrow (e^{i\theta}\zeta_1, e^{2i\theta}\zeta_2) \text{ for } 0 \leq \theta < 2\pi$$

Z₂(κ): $x \rightarrow -x$

$$\Leftrightarrow (\zeta_1, \zeta_2) \rightarrow (\bar{\zeta}_1, \bar{\zeta}_2)$$

O(2) 不変多項式:

$$\mathcal{P}_1 = \mathcal{P}_1(u, v, w, \lambda), \quad \mathcal{P}_2 = \mathcal{P}_2(u, v, w, \lambda)$$

$$\mathcal{Q}_1 = \mathcal{Q}_1(u, v, w, \lambda), \quad \mathcal{Q}_2 = \mathcal{Q}_2(u, v, w, \lambda)$$

$$u = |\zeta_1|^2, \quad v = |\zeta_2|^2, \quad w = \bar{\zeta}_1^2\zeta_2 + \zeta_1^2\bar{\zeta}_2$$

O(2)-同変ベクトル場によって生成された振幅方程式:

$$\dot{\zeta}_1 = \zeta_1 \mathcal{P}_1 + \bar{\zeta}_1 \zeta_2 \mathcal{Q}_1, \quad \dot{\zeta}_2 = \zeta_2 \mathcal{P}_2 + \zeta_1^2 \mathcal{Q}_2$$

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定常解 $r_1 \neq 0, r_2 \neq 0$, and $\cos \Theta = \pm 1$: 混合モード

$$0 = \sigma_1 r_1 \pm \beta_1 r_1 r_2 + (\kappa_1 r_1^2 + \kappa_2 r_2^2) r_1,$$

$$0 = \sigma_2 r_2 \pm \beta_2 r_1^2 + (\kappa_3 r_1^2 + \kappa_4 r_2^2) r_2$$

定常解 $r_1 \neq 0, r_2 \neq 0$, and $\cos \Theta \neq \pm 1$: 伝播波

$$0 = \sigma_1 r_1 + \beta_1 r_1 r_2 \cos \Theta + (\kappa_1 r_1^2 + \kappa_2 r_2^2) r_1,$$

$$0 = \sigma_2 r_2 + \beta_2 r_1^2 \cos \Theta + (\kappa_3 r_1^2 + \kappa_4 r_2^2) r_2,$$

$$0 = \beta_2 r_1^2 r_2^{-1} + 2\beta_1 r_2$$

$$\theta_2 - 2\theta_1 = \Theta \neq n\pi$$

$$\dot{\theta}_1 = \beta_1 r_2 \sin \Theta \rightarrow \theta_1 = (\beta_1 r_2 \sin \Theta) t + \theta_1(0), \quad \theta_2 = 2\theta_1 + \Theta$$

$$\psi = r_1 \phi_1(z) e^{i(k_0x + \theta_1)} + r_2 \phi_2(z) e^{2i(k_0x + \theta_1 + \Theta)} + \dots$$

$$e^{i(k_0x + \theta_1)} \propto e^{i\{k_0(x - [-k_0^{-1}\beta_1 r_2 \sin \Theta] \cdot t)\}}$$

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Relative Equilibrium

原点におけるテイラー展開:

$$\dot{\zeta}_1 = \sigma_1 \zeta_1 + \beta_1 \bar{\zeta}_1 \zeta_2 + (\kappa_1 |\zeta_1|^2 + \kappa_2 |\zeta_2|^2) \zeta_1,$$

$$\dot{\zeta}_2 = \sigma_2 \zeta_2 + \beta_2 \zeta_1^2 + (\kappa_3 |\zeta_1|^2 + \kappa_4 |\zeta_2|^2) \zeta_2$$

$$\zeta_1 = r_1(t) e^{i\theta_1(t)}, \quad \zeta_2 = r_2(t) e^{i\theta_2(t)}$$

$$\dot{r}_1 = \sigma_1 r_1 + \beta_1 r_1 r_2 \cos \Theta + (\kappa_1 r_1^2 + \kappa_2 r_2^2) r_1,$$

$$\dot{r}_2 = \sigma_2 r_2 + \beta_2 r_1^2 \cos \Theta + (\kappa_3 r_1^2 + \kappa_4 r_2^2) r_2,$$

$$\dot{\Theta} = -(\beta_2 r_1^2 r_2^{-1} + 2\beta_1 r_2) \sin \Theta$$

where $\Theta(t) = \theta_2(t) - 2\theta_1(t)$.

定常解 $r_1 = 0, r_2 \neq 0$: 単純モード

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上下の反転対称性 Z₂: $(\zeta_1, \zeta_2) \rightarrow (-\zeta_1, -\zeta_2)$

O(2) ⊕ Z₂-不変多項式:

$$\mathcal{P}_1 = \mathcal{P}_1(u, v, w^2, \lambda), \quad \mathcal{P}_2 = \mathcal{P}_2(u, v, w^2, \lambda)$$

$$\mathcal{Q}_1 = \mathcal{Q}_1(u, v, w^2, \lambda), \quad \mathcal{Q}_2 = \mathcal{Q}_2(u, v, w^2, \lambda)$$

$$u = |\zeta_1|^2, \quad v = |\zeta_2|^2, \quad w = \bar{\zeta}_1^2 \zeta_2 + \zeta_1^2 \bar{\zeta}_2$$

O(2) ⊕ Z₂-同変ベクトル場によって生成された振幅方程式:

$$\dot{\zeta}_1 = \zeta_1 \mathcal{P}_1 + \bar{\zeta}_1 \zeta_2 w \mathcal{Q}_1, \quad \dot{\zeta}_2 = \zeta_2 \mathcal{P}_2 + \zeta_1^2 w \mathcal{Q}_2$$

原点におけるテイラー展開:

$$\dot{\zeta}_1 = \sigma_1 \zeta_1 + (\kappa_1 u + \mu_1 v) \zeta_1 + \chi_1 \bar{\zeta}_1 \zeta_2 w + (\lambda_{11} u^2 + \lambda_{12} uv + \lambda_{13} v^2) \zeta_1,$$

$$\dot{\zeta}_2 = \sigma_2 \zeta_2 + (\kappa_2 u + \mu_2 v) \zeta_2 + \chi_2 \zeta_1^2 w + (\lambda_{21} u^2 + \lambda_{22} uv + \lambda_{23} v^2) \zeta_2$$

$\zeta_1 \neq 0, \zeta_2 = 0$: 大きなロール, $\zeta_1 = 0, \zeta_2 \neq 0$: 小さなロール

$\zeta_1 \neq 0, \zeta_2 \neq 0$: 混合ロール

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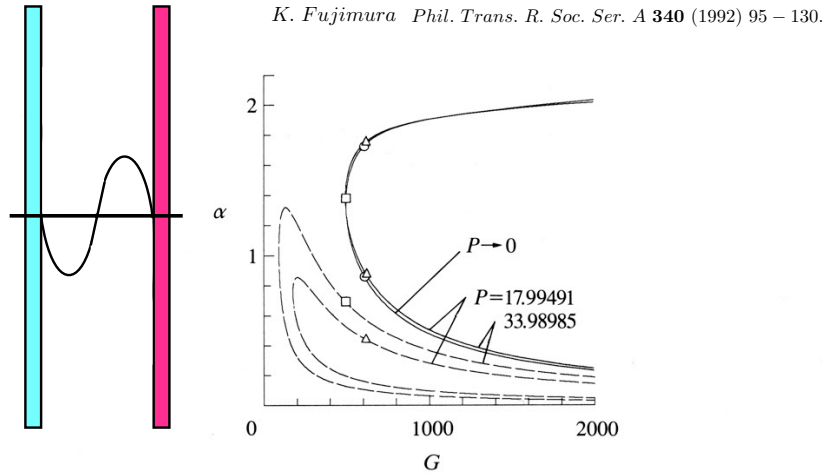


Figure 1. Neutral stability curves for different Prandtl numbers, $P \rightarrow 0$, $P = 17.99491$, and 33.98985 . Solid curves denote the neutral curves for stationary modes while dashed curves denote those for travelling modes. Open circles denote parameter sets for 1:2 resonance, open squares denote those for 1:1:2 resonance, and open triangles denote those for 1:1:2:4 resonance.

O(2) 対称性の下での 1 : 2 共鳴による構造不安定なヘテロクリニック軌道

- Armbruster, Guckenheimer & Holmes, *Physica* 29D (1988) pp.257–282.
Proctor & Jones, *J.Fluid Mech.* 188 (1988) pp.301–355.
Porter & Knobloch, *Physica* 159D (2001) pp.125–154.
O(2) symmetric case under periodic boundary conditions.

$$\begin{aligned} \dot{z}_1 &= \mu_1 z_1 + \alpha \bar{z}_1 z_2 + z_1 (d_{11}|z_1|^2 + d_{12}|z_2|^2), \\ \dot{z}_2 &= \mu_2 z_2 + \beta z_1^2 + z_2 (d_{21}|z_1|^2 + d_{22}|z_2|^2) \end{aligned}$$

- Porter & Knobloch, *Physica* 201D (2005) pp.318–344.
Slightly broken symmetry: $O(2) \rightarrow SO(2)$

$$\begin{aligned} \dot{z}_1 &= (\mu_1 + i\epsilon\omega_1) z_1 + \alpha \bar{z}_1 z_2 + z_1 (d_{11}|z_1|^2 + d_{12}|z_2|^2), \\ \dot{z}_2 &= (\mu_2 + i\epsilon\omega_2) z_2 + \beta z_1^2 + z_2 (d_{21}|z_1|^2 + d_{22}|z_2|^2) \end{aligned}$$

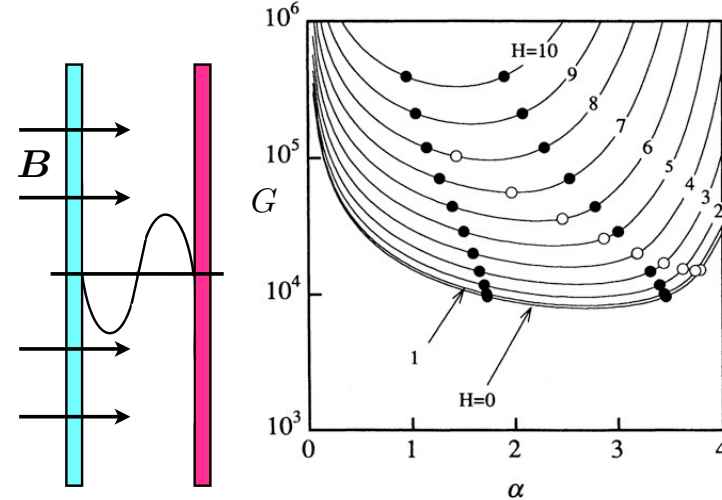


Fig. 2. Linear neutral stability curves. The open circle indicates the parameter set at which the nonlinear degenerate situation occurs where the cubic coefficient of the Stuart–Landau equation (the first Landau constant) for a monochromatic mode vanishes. The closed circles denote the 1:2 interaction points.

O(2) 対称性の下での 1 : 2 共鳴による構造不安定なヘテロクリニック軌道

- Cox, *Physica* 95D (1996) pp.50–61.
A long-wave PDE model + amplitude equations under O(2)

$$\frac{\partial \theta}{\partial t} = -\alpha \theta - \frac{R - R_0}{R_0} \frac{\partial^2 \theta}{\partial x^2} - a \frac{\partial^4 \theta}{\partial x^4} + b \frac{\partial}{\partial x} \left(\frac{\partial \theta}{\partial x} \right)^3 + c \frac{\partial^2}{\partial x^2} \left(\frac{\partial \theta}{\partial x} \right)^2$$

- Mercader, Prat & Knobloch, *Int.J.Bifurcation and Chaos* 12 (2002) pp.2501–22.
Rayleigh–Bénard convection without midplane reflection symmetry.
- Nore, Moisy & Quartier, *Phys.Fluids* (2005) 17 064103.
von Kármán swirling flow, laboratory experiment

対称群によって互いに等価である C^n 級ベクトル空間において、構造安定なヘテロクリニックサイクルが存在.

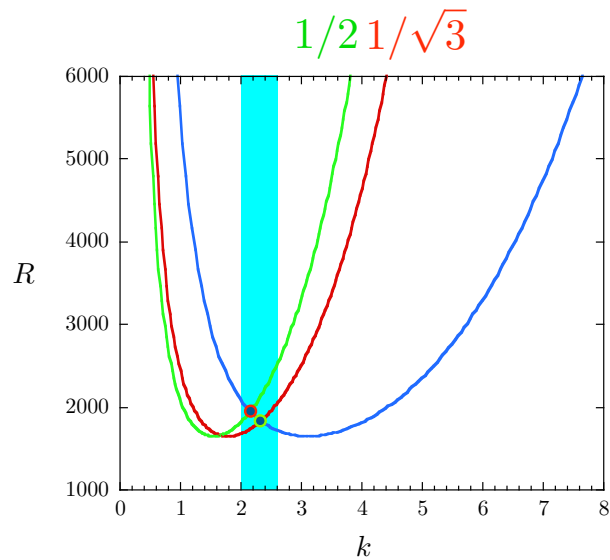
Busse & Heikes: *Science* **208** (1980) pp.173–175, Guckenheimer & Holmes (1988): *Math. Proc. Camb. Phil. Soc.* (1988) **103** pp.189–192

$$\dot{x} = x(l + ax^2 + by^2 + cz^2),$$

$$\dot{y} = y(l + ay^2 + bz^2 + cx^2),$$

$$\dot{z} = z(l + az^2 + bx^2 + cy^2).$$

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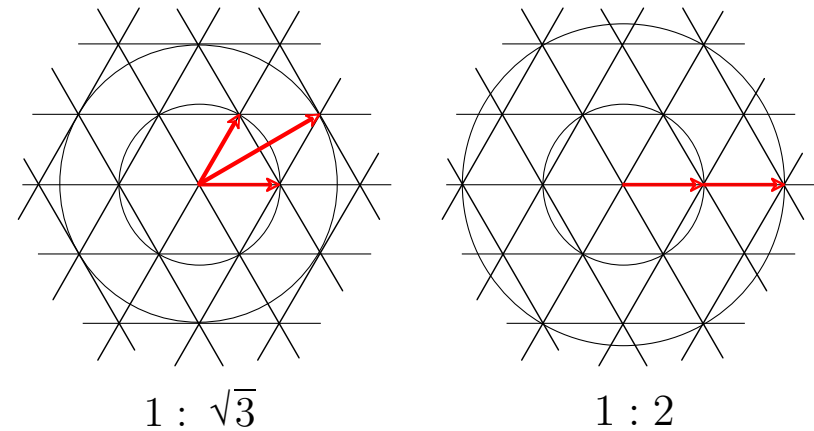
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Theorem (Guckenheimer & Holmes):

In the space of one-parameter families of G -equivariant vector fields, there is an open set V with the following property: If $X_\lambda \in V$, then there is a bifurcation of the origin at $\lambda = \lambda_0$ such that

- (a) the origin loses stability as λ increases through λ_0
- (b) for $\lambda > \lambda_0$, there is a stable heteroclinic cycle near the origin.

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Daumont, I., Kassner, K., Misbach, C. & Valance, A. 1997 Cellular self-propulsion of two-dimensional dissipative structures and spatial-period tripling Hopf bifurcation. *Phys. Rev. E* **55**, 6902–6906. (doi:10.1103/PhysRevE.55.6902)

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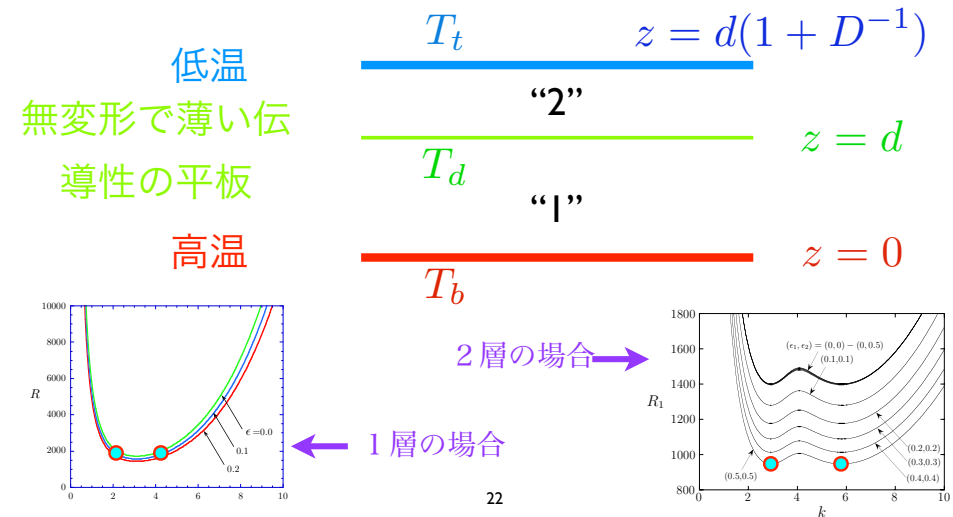
問題設定 -- 2層RB問題

The 1 : 2 spatial resonance on a hexagonal lattice in two-layered Rayleigh–Bénard problems

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1. Two-layered Rayleigh–Bénard problem

$$\rho_0^{(j)} \frac{D\vec{v}_j^*}{Dt^*} = -\nabla^* p_j^* - \rho_0^{(j)} g [1 - \alpha_1^{(j)} (T_j^* - T_d) - \alpha_2^{(j)} (T_j^* - T_d)^2] \mathbf{e}_z + \mu_j \Delta^* \vec{v}_j^*,$$

$$\frac{DT_j^*}{Dt^*} = \kappa_j \Delta^* T_j^*, \quad \nabla^* \cdot \vec{v}_j^* = 0, \quad (j = 1, 2). \quad (1.1)$$

\mathbf{e}_z : the unit vector upward in the z -direction,
 g : the acceleration due to the gravity,
 μ_1, μ_2 : the viscous coefficients,
 κ_1, κ_2 : the thermal diffusivities,
 $\rho_0^{(1)}, \rho_0^{(2)}$: the densities.
 physical properties: evaluated at $T_1^* = T_d$ and $T_2^* = T_d$.

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2. Non-dimensionalization

$$t^* = \frac{d^2}{\kappa_1} t, \quad \vec{v}_j^* = \frac{\kappa_j}{d} \vec{v}_j, \quad \vec{x}^* = d\vec{x},$$

$$p_j^* = -d\rho_0^{(j)} g \int^z [1 - \alpha_1^{(j)} \tilde{D}_j (T^{(j)} - T_d)(1-z) - \alpha_2^{(j)} \tilde{D}_j^2 (T^{(j)} - T_d)^2 (1-z)^2] dz + \rho_0^{(j)} \frac{\kappa_j^2}{d^2} \pi_1,$$

$$T_j^* - T_d = (T^{(j)} - T_d) \tilde{D}_j [(1-z) + \theta_j(x, y, z; t)], \quad (j = 1, 2) \quad (2.1)$$

where we set $\tilde{D}_j = (-D)^{j-1}$, $T^{(1)} = T_b$, and $T^{(2)} = T_t$.

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Assume $\kappa_1 = \kappa_2$ and $\bar{T} = 1 - z$. In the non-dimensional form:

$$P_j^{-1} \frac{D\vec{v}_j}{Dt} = -P_j^{-1} \nabla \pi_j + R_j K_j \theta_j \mathbf{e}_z + R_j K_j \epsilon_j (2\bar{T}\theta_j + \theta_j^2) \mathbf{e}_z + \Delta \vec{v}_j,$$

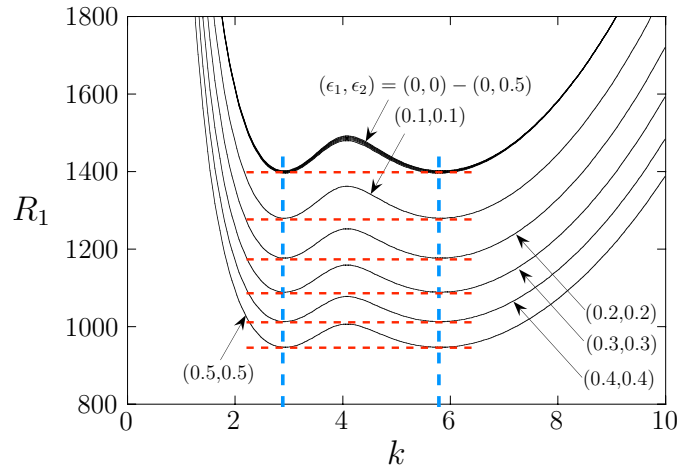
$$\frac{D\theta_j}{Dt} - w_j = \Delta \theta_j, \quad \nabla \cdot \vec{v}_j = 0, \quad (j = 1, 2), \quad \vec{v}_j = (u_j, v_j, w_j)^T \quad (2.2)$$

where

$$R_j = \frac{\rho_0^{(j)} g \alpha_1^{(j)} (T^{(j)} - T_d) d^3}{\tilde{D}_j^3 \mu_1 \kappa_1}, \quad P_j = \frac{\nu_j}{\kappa_j}, \quad K_j = \tilde{D}_j^4, \quad \epsilon_j = \frac{\alpha_2^{(j)} (T^{(j)} - T_d) \tilde{D}_j}{\alpha_1^{(j)}}. \quad (2.3)$$

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2層レイリー・ベナール問題に対する 中立曲線



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3. Boundary conditions

$$\vec{v}_1 = \vec{v}_2 = 0 \quad \text{at} \quad z = 0, 1, 1 + D^{-1}, \quad \theta_1 = \theta_2 = 0 \quad \text{at} \quad z = 0, 1 + D^{-1}. \quad (3.1)$$

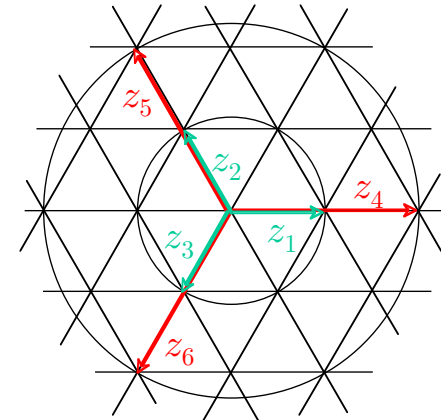
On the dividing plate, the temperatures need to satisfy

$$T_1^* = T_2^*, \quad \kappa_1 \frac{dT_1^*}{dz^*} = \kappa_2 \frac{dT_2^*}{dz^*} \quad \text{at} \quad z^* = d, \quad (3.2)$$

which yield

$$\theta_1 = \frac{R_2 D^4 \alpha_1^{(1)} \nu_2 \kappa_2}{R_1 \alpha_1^{(2)} \nu_1 \kappa_1} \theta_2 \equiv G \theta_2, \quad \frac{d\theta_1}{dz} = G \frac{d\theta_2}{dz} \quad \text{at} \quad z = 1. \quad (3.3)$$

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$$\psi(x, y, z, t) =$$

$$z_1(t) \phi_1^{(1)}(z) e^{ikx} + z_2(t) \phi_1^{(1)}(z) e^{ik(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)} + z_3(t) \phi_1^{(1)}(z) e^{ik(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y)} + c.c.$$

$$+ z_4(t) \phi_2^{(1)}(z) e^{2ikx} + z_5(t) \phi_2^{(1)}(z) e^{2ik(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)} + z_6(t) \phi_2^{(1)}(z) e^{2ik(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y)} + c.c.$$

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同変ベクトル場

$$\mathbf{z} = g(\mathbf{z}, \gamma)$$

$$\mathbf{z} = (z_1(t), z_2(t), z_3(t), z_4(t), z_5(t), z_6(t)) \in \mathbb{C}^6$$

$$\mathbf{g} = (g_1(\mathbf{z}), g_2(\mathbf{z}), g_3(\mathbf{z}), g_4(\mathbf{z}), g_5(\mathbf{z}), g_6(\mathbf{z}))$$

対称群 Γ の作用の下での同変性：

$$g(\gamma \mathbf{z}) = \gamma g(\mathbf{z}) \text{ for all } \gamma \in \Gamma$$

ここで、正六角形の持つ対称群： $\Gamma = D_6 \dagger T^2$

Γ – equivariant polynomials $g(\mathbf{z})$:

$$g(\gamma \mathbf{z}) = \gamma g(\mathbf{z}) \text{ for all } \gamma \in \Gamma$$

(b1) (b2) (b3) (b4) (b5) (b6) (b7) (b8) (b9) (b10) (b11) (b12)

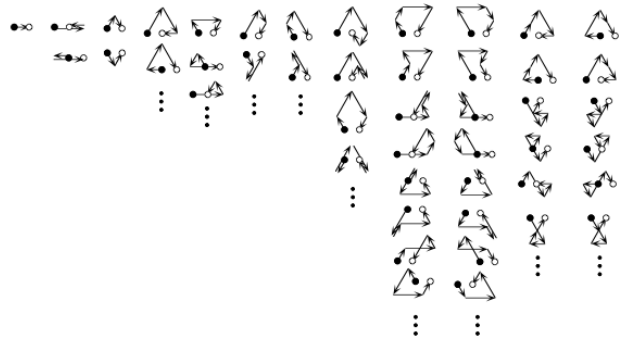


Figure A2. Open paths on a hexagonal lattice. Closed circles denote the starting point $(0, 0)$ and open circles denote the ending point $(k_c, 0)$. (b1): z_1 , (b2): $\bar{z}_1 z_4$, (b3): $\bar{z}_2 \bar{z}_3$, (b4): $\bar{z}_1 \bar{z}_5 \bar{z}_6$, (b5): $z_2 z_3 z_4$, (b6): $\bar{z}_2 z_3 \bar{z}_6$, (b7): $z_2 \bar{z}_3 \bar{z}_5$, (b8): $z_2 z_3 \bar{z}_5 \bar{z}_6$, (b9): $z_2 \bar{z}_3 z_4 z_6$, (b10): $\bar{z}_2 z_3 z_4 z_5$, (b11): $\bar{z}_1 \bar{z}_3^2 \bar{z}_5$, (b12): $\bar{z}_1 \bar{z}_2^2 \bar{z}_6$. See (A 5).

$$\begin{array}{ll} R_{2\pi/3} & \mathbf{z} \rightarrow (z_2 z_3, z_1, z_5, z_6, z_4) \\ c_v & \mathbf{z} \rightarrow (z_1, z_3, z_2, z_4, z_6, z_5) \\ c & \mathbf{z} \rightarrow (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4, \bar{z}_5, \bar{z}_6) \\ (s, t) \in T^2 & \mathbf{z} \rightarrow (e^{is} z_1, e^{-i(s+t)} z_2, e^{it} z_3, e^{2is} z_4, e^{-2i(s+t)} z_5, e^{2it} z_6) \end{array} \quad \Gamma = D_6 \dagger T^2$$

$$f(\gamma \mathbf{z}) = f(\mathbf{z}) \text{ for all } \gamma \in \Gamma$$

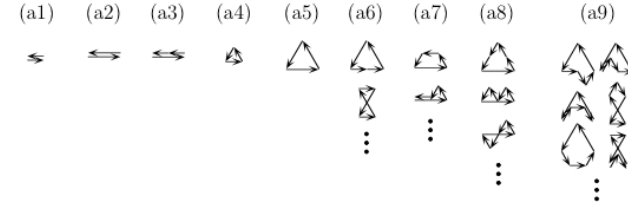


Figure A1. Elements of closed polygons on a hexagonal lattice in the wavenumber plane. Short arrows denote wavevectors with k_c and long arrows denote wavevectors with $2k_c$. (a1): $|z_1|^2$, (a2): $|z_4|^2$, (a3): $\bar{z}_1^2 z_4$, (a4): $z_1 z_2 z_3$, (a5): $z_4 z_5 z_6$, (a6): $z_1^2 z_5 z_6$, (a7): $\bar{z}_1 z_2 z_3 z_4$, (a8): $z_2^2 z_3^2 z_4$, (a9): $z_1 \bar{z}_2 \bar{z}_3 z_5 z_6$. See (A 5).

Γ – equivariant polynomials $g(\mathbf{z})$:

$$g(\gamma \mathbf{z}) = \gamma g(\mathbf{z}) \text{ for all } \gamma \in \Gamma$$

(c1) (c2) (c3) (c4) (c5) (c6) (c7) (c8) (c9)

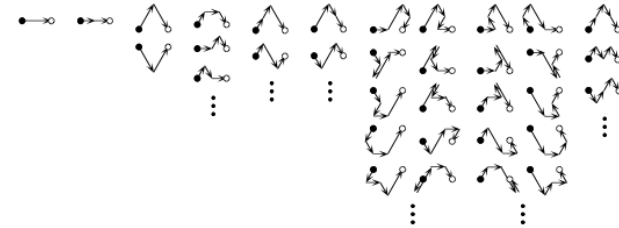


Figure A3. Open paths on a hexagonal lattice. Closed circles denote the starting point $(0, 0)$ and open circles denote the ending point $(2k_c, 0)$. (c1): z_4 , (c2): z_4^2 , (c3): $\bar{z}_5 \bar{z}_6$, (c4): $z_1 \bar{z}_2 \bar{z}_3$, (c5): $\bar{z}_3^2 \bar{z}_5$, (c6): $\bar{z}_2^2 \bar{z}_6$, (c7): $z_1 \bar{z}_2 z_3 \bar{z}_6$, (c8): $z_1 z_2 \bar{z}_3 \bar{z}_5$, (c9): $\bar{z}_2^2 \bar{z}_3^2$. See (A 5).

Amplitude equation generated by Γ – equivariant vector field

$$\dot{z}_j = g_j(z_1, z_2, z_3, z_4, z_5, z_6) \text{ for } 1 \leq j \leq 6$$

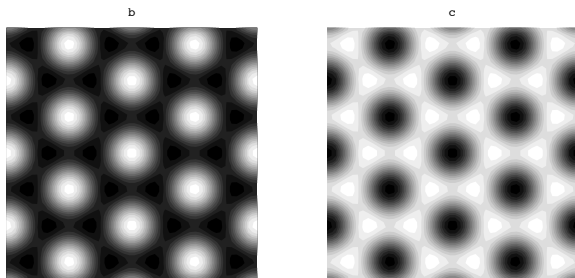
$$\begin{aligned}
 g_1(z) &= P_1 z_1 + P_2 z_1 z_4 + P_3 z_2 z_3 + P_4 z_1 z_5 z_6 + P_5 z_2 z_3 z_4 \\
 &\quad + P_6 (z_2 z_3 z_6 + z_2 z_3 z_5) + P_7 (z_2 z_3 z_5 z_6 + z_2 z_3 z_4 z_6) \\
 &\quad + P_8 z_2 z_3 z_4 z_5 + P_9 (z_1 z_3^2 z_5 z_6 + z_1 z_2^2 z_6) \\
 g_4(z) &= Q_1 z_4 + Q_2 z_1^2 + Q_3 z_5 z_6 + Q_4 z_1 z_2 z_3 + Q_5 (z_3^2 z_5 + z_2^2 z_6) \\
 &\quad + Q_6 z_2^2 z_3^2 + Q_7 (z_1 z_2 z_3 z_6 + z_1 z_2 z_3 z_5)
 \end{aligned}$$

P_j, Q_j : real functions of invariant polynomials

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単純モード (正六角形): $(0, 0, 0, x, x, x)$, $x \in \mathbb{R}$ $D_6 + Z_2^2$
 $\{\mathbb{R}_{2\pi/3}, c, c_v, Z_2(\pi, 0), Z_2(0, \pi)\} \mathbb{R}\{(0, 0, 0, 1, 1, 1)\}$

$$\sigma_2 + \delta_2 x + (\mu_{21} + 2\mu_{22})x^2 = 0$$



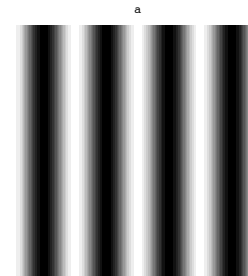
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定常解

自明解: $(0, 0, 0, 0, 0, 0)$ $D_6 + T^2$
 $\{\mathbb{R}_{2\pi/3}, c, c_v, S^1(\theta, 0), S^1(0, \theta)\}$

単純モード (ロール): $(0, 0, 0, x, 0, 0)$, $x \in \mathbb{R}$ $S^1 + Z_2^3$
 $\{c, c_v, Z_2(\pi, 0), S^1(0, \theta)\} \mathbb{R}\{(0, 0, 0, 1, 0, 0)\}$

$$\sigma_2 + \mu_{21}x^2 = 0$$

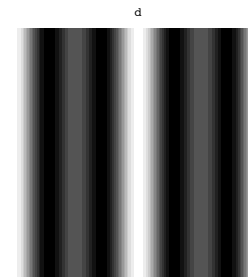


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混合モード (混合ロール): $(x, 0, 0, y, 0, 0)$, $x, y \in \mathbb{R}$ $S^1 + Z_2^2$
 $\{c, c_v, S^1(0, \theta)\} \mathbb{R}\{(1, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0)\}$

$$\sigma_1 + \beta_1 y + \kappa_{11}x^2 + \mu_{11}y^2 = 0,$$

$$\sigma_2 y + \beta_2 x^2 + \kappa_{21}x^2 y + \mu_{21}y^3 = 0$$

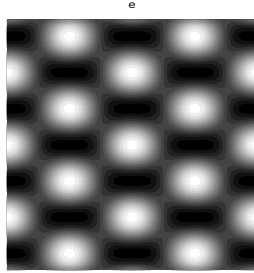


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単純モード (長方形): $(0,0,0,x,y,y)$, $x \neq y \in \mathbb{R} \mathbb{Z}_2^4$
 $\{c, c_v, Z_2(\pi, 0), Z_2(0, \pi)\} \mathbb{R}\{(0,0,0,1,0,0), (0,0,0,0,1,1)\}$

$$\sigma_2 x + \delta_2 y^2 + (\mu_{21} x^2 + 2\mu_{22} y^2) x = 0,$$

$$\sigma_2 + \delta_2 x + \mu_{22} x^2 + (\mu_{21} + \mu_{22}) y^2 = 0$$



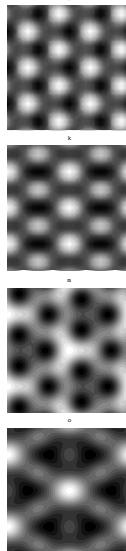
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単純モード (三角形): $(0,0,0,z,z,z)$, $z \in \mathbb{C} \mathbb{D}_3 + \mathbb{Z}_2^2$
 $\{\mathbb{R}_{2\pi/3}, c_v, Z_2(\pi, 0), Z_2(0, \pi)\} \mathbb{C}\{(0,0,0,1,1,1)\}$

混合モード (長方形ロール): $(z,0,0,x,y,y)$, $z, x \neq y \in \mathbb{R} \mathbb{Z}_2^3$
 $\{c, c_v, Z_2(0, \pi)\} \mathbb{R}\{(1,0,0,0,0,0), (0,0,0,1,0,0)\}$

混合モード (混合三角形): (x,x,x,y,y,y) , $x, y \in \mathbb{C} \mathbb{D}_3$
 $\{\mathbb{R}_{2\pi/3}, c_v\} \mathbb{C}\{(1,1,1,0,0,0), (0,0,0,1,1,1)\}$

混合モード (混合長方形): (x,y,y,u,v,v) , $x \neq y, u \neq v \in \mathbb{R} \mathbb{Z}_2^2$
 $\{c, c_v\} \mathbb{R}\{(1,0,0,0,0,0), (0,1,1,0,0,0)\}$

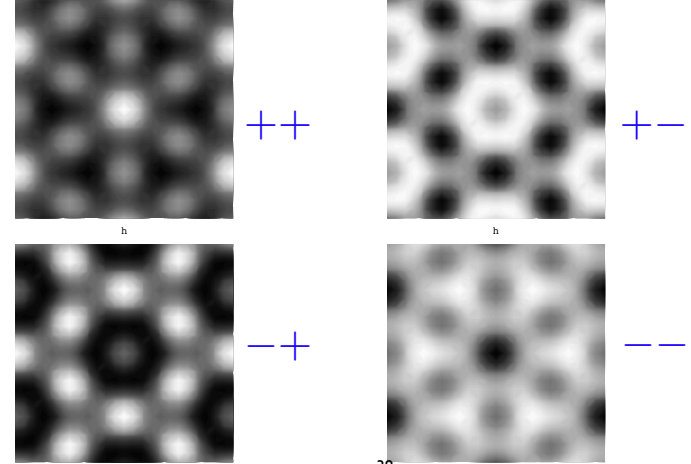


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混合モード (混合六角形): (x,x,x,y,y,y) , $x, y \in \mathbb{R} \mathbb{D}_6$
 $\{\mathbb{R}_{2\pi/3}, c, c_v\} \mathbb{R}\{(1,1,1,0,0,0), (0,0,0,1,1,1)\}$

$$\sigma_1 + \beta_1 y + \delta_1 x + (\kappa_{11} + 2\kappa_{12}) x^2 + (\mu_{11} + 2\mu_{12} + \nu_1) y^2 + (2\eta_1 + \xi_1) xy = 0,$$

$$\sigma_2 y + \beta_2 x^2 + \delta_2 y^2 + (\kappa_{21} + 2\kappa_{22} + 2\xi_2) x^2 y + (\mu_{21} + 2\mu_{22}) y^3 + \nu_2 x^3 = 0$$



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$\dot{z} = g(z, \lambda)$ の不動点 z_0 の軌道安定性

$$x_j = \operatorname{Re} z_j, y_j = \operatorname{Im} z_j, g_j^r = \operatorname{Re} g_j, g_j^i = \operatorname{Im} g_j, 1 \leq j \leq 6.$$

$$g(\gamma z) = \gamma g(z) \Rightarrow dg(\gamma z_0, \lambda) \gamma = \gamma dg(z_0, \lambda) \text{ for all } \gamma \in \Gamma.$$

$$\Leftrightarrow dg(z_0, \lambda) \sigma = \sigma dg(z_0, \lambda) \text{ for all } \sigma \in \Sigma_z$$

ここで Σ_z : 次で定義される固定化部分群

$$\Sigma_z = \{\sigma \in \Gamma : \sigma z = z\} \text{ for } z \in \mathbb{C}^6$$

$\gamma(\theta)$ を Γ において $\gamma(0) = 1$ を満たすなめらかな曲線であるとする.

$$g(z_0, \lambda) = 0 \text{ and } \gamma g(z) = g(\gamma z) \Rightarrow g(\gamma(\theta) z, \lambda) = 0$$

$$dg(z_0) \left(\frac{\partial \gamma}{\partial \theta} \right)_{\theta=0} z_0 = 0.$$

$$\Rightarrow \left(\frac{\partial \gamma}{\partial \theta} \right)_{\theta=0} z_0: \text{ゼロ固有値に属する } dg(z_0) \text{ の固有ベクトル}$$

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$Z_2(\pi, 0) :$

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & a_{24} & a_{24} & a_{14} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{42} & a_{44} & a_{45} & a_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{42} & a_{45} & a_{44} & a_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{52} & a_{45} & a_{45} & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & d_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{21} & d_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & d_{24} & d_{24} & d_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{42} & d_{44} & d_{45} & d_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{42} & d_{45} & d_{44} & d_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{52} & d_{45} & d_{45} & d_{44} \end{pmatrix}$$

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$Z_2(0, \pi) :$

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{45} & a_{44} & a_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{45} & a_{45} & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{44} & d_{45} & d_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{45} & d_{44} & d_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{45} & d_{45} & d_{44} \end{pmatrix}$$

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$$S^1(s, 0) = (e^{is}z_1, e^{-is}z_2, z_3, e^{2is}z_4, e^{-2is}z_5, z_6),$$

$$S^1(0, t) = (z_1, e^{-it}z_2, e^{it}z_3, z_4, e^{-2it}z_5, e^{2it}z_6)$$

$$\frac{\partial \rho(s, 0)}{\partial s} z = (iz_1, -iz_2, 0, 2iz_4, -2iz_5, 0) \Rightarrow (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1, 0)$$

$$\frac{\partial \rho(0, t)}{\partial t} z = (0, -iz_2, z_3, 0, -2iz_5, 2iz_6) \Rightarrow (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -1, 1)$$

$$(dg) \cdot \ker(dg) = 0$$

$$d_{44} - d_{45} = 0, \quad d_{45} - d_{44} = 0, \quad d_{45} - d_{45} = 0,$$

$$-d_{45} + d_{45} = 0, \quad -d_{44} + d_{45} = 0, \quad -d_{45} + d_{44} = 0$$

yields $d_{44} = d_{45}$.

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} & a_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{45} & a_{44} & a_{45} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{45} & a_{45} & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{44} & d_{44} & d_{44} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{44} & d_{44} & d_{44} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_{44} & d_{44} & d_{44} \end{pmatrix}$$

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単純正六角形に対する固有値

$$\frac{\partial g_4^r}{\partial x_4} + 2\frac{\partial g_4^r}{\partial x_5}, 3\frac{\partial g_4^i}{\partial y_4}, 0(2), \frac{\partial g_4^r}{\partial x_4} - \frac{\partial g_4^r}{\partial x_5}(2), \frac{\partial g_1^r}{\partial x_1}(3) \text{ and } \frac{\partial g_1^i}{\partial y_1}(3)$$

単純口ールに対する固有値

$$0, \frac{\partial g_1^r}{\partial x_1}, \frac{\partial g_1^i}{\partial y_1}, \frac{\partial g_4^r}{\partial x_4}, \frac{\partial g_5^r}{\partial x_5} + \frac{\partial g_5^r}{\partial x_6}(2), \frac{\partial g_5^r}{\partial x_5} - \frac{\partial g_5^r}{\partial x_6}(2), \text{ and } \frac{\partial g_2^r}{\partial x_2}(4)$$

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$\sigma^{(1)}$ に属する固有関数 $\psi^{(1)}(x, y, z)$ は以下に比例する :

$$e^{\pm ikx}, e^{\pm ik(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)}, e^{\pm ik(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y)}, \\ e^{\pm 2ikx}, e^{\pm 2ik(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)}, e^{\pm 2ik(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y)}$$

$$E_1 = e^{ikx}, E_2 = e^{ik(-\frac{1}{2}x + \frac{\sqrt{3}}{2}y)}$$

とおき, $\psi^{(j)}$ をフーリエ分解 :

$$\psi^{(j)} = \sum_{m,n} \phi_{mn}^{(j)} E_1^m E_2^n.$$

$$L_{mn}(\mu) \phi_{mn}^{(j)} = \sigma_{mn}^{(j)} S_{mn} \phi_{mn}^{(j)},$$

$$S_{mn} = \mathcal{S}|_{\partial_x \rightarrow i(m-\frac{n}{2})\gamma, \partial_y \rightarrow i\frac{\sqrt{3}}{2}n\gamma, \partial_z \rightarrow d/dz},$$

$$L_{mn} = \mathcal{L}|_{\partial_x \rightarrow i(m-\frac{n}{2})\gamma, \partial_y \rightarrow i\frac{\sqrt{3}}{2}n\gamma, \partial_z \rightarrow d/dz}.$$

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$\psi(x, y, z, t)$ に対する非線形偏微分方程式 :

$$\frac{\partial}{\partial t} \mathcal{S}\psi = \mathcal{L}(\mu)\psi + \mathcal{N}(\psi, \psi), \mathcal{H}\psi = 0 \text{ on } \partial D$$

$\mathcal{S}, \mathcal{L}, \mathcal{H}$: 線形作用素, \mathcal{N} : 2次の非線形項

線形固有値問題 :

$$\mathcal{L}(\mu)\psi^{(j)}(x, y, z) = \sigma^{(j)} \mathcal{S}\psi^{(j)}(x, y, z), \quad j \geq 1.$$

$$\text{Re } \sigma^{(1)} > \text{Re } \sigma^{(2)} > \text{Re } \sigma^{(3)} > \dots$$

$$\text{Re } \sigma^{(1)} = 0, \quad \text{Re } \sigma^{(j)} < 0 \text{ for } j \geq 2$$

を仮定すると ...

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随伴問題 :

$$\tilde{L}_{mn}(\mu) \tilde{\phi}_{mn}^{(j)} = \sigma_{mn}^{(j)} \tilde{S}_{mn} \tilde{\phi}_{mn}^{(j)},$$

with

$$\langle \tilde{\phi}_{mn}^{(j)}, (L_{mn}(\mu) - \sigma_{mn}^{(k)} S_{mn}) \phi_{mn}^{(k)} \rangle = \langle \phi_{mn}^{(j)}, (\tilde{L}_{mn}(\mu) - \sigma_{mn}^{(j)} \tilde{S}_{mn}) \tilde{\phi}_{mn}^{(j)} \rangle.$$

$$\psi(x, y, z, t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{j=1}^{\infty} A_{mn}^{(j)}(t) \phi_{mn}^{(j)} E_1^m E_2^n, \quad A_{-m-n}^{(j)} = \bar{A}_{mn}^{(j)}$$

$$\dot{A}_{mn}^{(j)} = \sigma_{mn}^{(j)}(\mu) A_{mn}^{(j)} + \sum_{k,l} \lambda_{k,l,m-k,n-l}^{(j,p,q)} A_{k,l}^{(p)} A_{m-k,n-l}^{(q)}$$

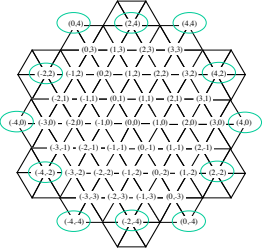
$$\sigma_{mn}^{(j)}(\mu) = \frac{\langle \tilde{\phi}_{mn}^{(j)}, L_{mn}(\mu) \phi_{mn}^{(j)} \rangle}{\langle \tilde{\phi}_{mn}^{(j)}, S_{mn} \phi_{mn}^{(j)} \rangle}, \quad \lambda_{k,l,m-k,n-l}^{(j,p,q)} = \frac{\langle \tilde{\phi}_{mn}^{(j)}, N(\phi_{kl}^{(p)}, \phi_{m-k,n-l}^{(q)}) \rangle}{\langle \tilde{\phi}_{mn}^{(j)}, S_{mn} \phi_{mn}^{(j)} \rangle}.$$

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中心多様体低減

$$A_{mn}^{(j)} = h_{mn}^{(j)} (A_{\pm 1,0}^{(1)}, A_{0,\pm 1}^{(1)}, A_{\mp 1,\mp 1}^{(1)}, A_{\pm 2,0}^{(1)}, A_{0,\pm 2}^{(1)}, A_{\mp 2,\mp 2}^{(1)}).$$

$$h_{mn}^{(j)} = \sum_{k_1, k_2, l_1, l_2} \gamma_{k_1 k_2 l_1 l_2}^{(j)} A_{k_1, k_2}^{(1)} A_{l_1, l_2}^{(1)} + O(3).$$



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$$\gamma_{k_1 k_2 l_1 l_2}^{(j)} = \frac{\lambda_{k_1 k_2 l_1 l_2}^{(j11)}}{\sigma_{k_1 k_2}^{(1)} + \sigma_{l_1 l_2}^{(1)} - \sigma_{k_1 + k_2, l_1 + l_2}^{(j)}}$$

$$A_{10}^{(1)} \rightarrow z_1, A_{01}^{(1)} \rightarrow z_2, A_{-1-1}^{(1)} \rightarrow z_3,$$

$$A_{20}^{(1)} \rightarrow z_4, A_{02}^{(1)} \rightarrow z_5, \text{ and } A_{-2-2}^{(1)} \rightarrow z_6$$

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$$\begin{aligned} \dot{z}_1 = & \sigma_1 z_1 + \delta_1 \bar{z}_2 \bar{z}_3 + \beta_1 \bar{z}_1 z_4 + [\kappa_{11} |z_1|^2 + \kappa_{12} (|z_2|^2 + |z_3|^2)] z_1 \\ & + [\mu_{11} |z_4|^2 + \mu_{12} (|z_5|^2 + |z_6|^2)] z_1 + \nu_1 \bar{z}_1 \bar{z}_5 \bar{z}_6 + \xi_1 z_2 z_3 z_4 + \eta_1 (\bar{z}_2 z_3 \bar{z}_6 + z_2 \bar{z}_3 \bar{z}_5), \end{aligned}$$

$$\begin{aligned} \dot{z}_4 = & \sigma_2 z_4 + \delta_2 \bar{z}_5 \bar{z}_6 + \beta_2 z_1^2 + [\kappa_{21} |z_1|^2 + \kappa_{22} (|z_2|^2 + |z_3|^2)] z_4 \\ & + [\mu_{21} |z_4|^2 + \mu_{22} (|z_5|^2 + |z_6|^2)] z_4 + \nu_2 z_1 \bar{z}_2 \bar{z}_3 + \xi_2 (\bar{z}_3^2 \bar{z}_5 + \bar{z}_2^2 \bar{z}_6). \end{aligned}$$

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